Dirac Operator on a disk with global boundary conditions.*

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We compute the functional determinant for a Dirac operator in the presence of an Abelian gauge field on a bidimensional disk, under global boundary conditions of the type introduced by Atiyah-Patodi-Singer. We also discuss the connection between our result and the index theorem.

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I. INTRODUCTION

The wide application of functional determinants in Quantum and Statistical Physics is by now a well known fact. Typically, one is faced to the necessity of defining a regularized determinant for elliptic differential operators, among which the Dirac first order one plays a central role. An interesting related problem is the modification of physical quantities due to de presence of boundaries. The study of boundary effects has lately received much attention [1–9], since it is of importance in many different situations, like effective models for strong interactions , quantum cosmology and application of QFT to statistical systems , among others.

In previous work [10,11], we have studied elliptic Dirac boundary problems in the case of local boundary conditions. In particular, we have developed for this case a scheme for evaluating determinants from the knowledge of the associated Green's function, based on Seeley's theory of complex powers [12].

Another type of boundary conditions extensively studied in the literature are global ones, of the type introduced by Atiyah, Patodi and Singer (APS) [13] in connection with the index theorem for manifolds with boundaries (see [14] for a review.) Other motivations for considering these global (or spectral) conditions are their consistency with charge conjugation and chiral invariance, and the presence of topological obstructions for the chiral Dirac operator under local boundary conditions (although this restriction no longer holds when considering the whole Dirac operator [10].)

In this paper we present the complete evaluation of the determinant of the Dirac operator on a disk, in the presence of an axially symmetric Abelian flux, under spectral boundary conditions (see [15,16] and references therein for related work).

In section II we establish our conventions and set the problem to be considered.

In Section III we study zero modes and obtain the projector on the null space of the operator.

Section IV is devoted to the evaluation of the Green's function of the problem in the subspace orthogonal to such null space. The knowledge of this Green's function is necessary in the evaluation of the determinant, which is done in Section V. This evaluation is performed in two steps. The first one involves the consideration of the quotient of determinants with and without gauge field, under a fixed boundary condition. For this situation, we grow the field along a continuous path through a parameter α, and perform a point splitting regularization of the α-derivative of the logarithm of the quotient of determinants, written in terms of the Green's function.

The second step amounts to the evaluation, via ζ-function, of the quotient of the free operators, under different global boundary conditions, which is possible thanks to the explicit determination of their eigenfunctions and corresponding eigenvalues.

In Section VI we discuss the application of the APS index theorem to our example, with emphasis on the effect of the boundary isomorphism σ appearing in the Dirac operator.

Finally, Section VII contains our conclusions.

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II. SETTING OF THE PROBLEM

We will evaluate the determinant of the operator $D = i\partial\!\!\!/ + A\!\!\!/$ acting on functions defined on a two dimensional disk of radius R, under APS boundary conditions.

We take A_μ to be an Abelian gauge field. As it is well known, it can be written as $A_\mu = \epsilon_{\mu\nu}\ \partial_\nu\phi + \partial_\mu\eta$ $(\epsilon_{01} = -\epsilon_{10} = 1)$. We set $\eta \equiv 0$, thus choosing the Lorentz gauge. For ϕ we take a smooth bounded function $\phi = \phi(r)$; then $A_r = 0$ and $A_\theta(r) = -\partial_r\phi(r) = -\phi^{'}(r)$. We call

$$\kappa = \frac{\Phi}{2\pi} = \frac{1}{2\pi}\oint_{r=R} A_\theta\ R\ d\theta = -R\phi^{'}(R). \tag{1}$$

Our convention for two dimensional Dirac matrices is

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

which satisfy

$$\gamma_\mu\gamma_\nu\ = \delta_{\mu\nu}\ I\ + i\ \epsilon_{\mu\nu}\ \gamma_5. \tag{3}$$

Therefore, the free Dirac operator can be written as

$$i\ \partial\!\!\!/ = i\ (\gamma_0\ \partial_0 + \gamma_1\ \partial_1) = 2i\ \begin{pmatrix} 0 & \frac{\partial}{\partial X} \\ \frac{\partial}{\partial X^*} & 0 \end{pmatrix}, \tag{4}$$

where $X = x_0 + i\ x_1$ and $X^* = x_0 - i\ x_1$ or, in polar coordinates

$$i\ \partial\!\!\!/ = i(\gamma_r\ \partial_r + \frac{1}{r}\gamma_\theta\ \partial_\theta), \tag{5}$$

with

$$\gamma_r = e^{-i\gamma_5\theta}\gamma_0 = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \qquad \gamma_\theta = e^{-i\gamma_5\theta}\gamma_1 = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}. \tag{6}$$

With these conventions, the full Dirac operator can be written as

$$D = e^{-\gamma_5\phi(r)}\ i\ \partial\!\!\!/\ e^{-\gamma_5\phi(r)} = \begin{pmatrix} 0 & \sigma^{-1}(\partial_r + B) \\ \sigma\ (-\partial_r + B) & 0 \end{pmatrix}, \tag{7}$$

with

$$\sigma = -\ i\ e^{i\theta}, \tag{8}$$

and

$$B(r) = -\frac{i}{r}\ \partial_\theta +\ \partial_r\phi(r). \tag{9}$$

At the boundary, the self adjoint operator $B(r)$ becomes

$$B \equiv B(R) = -\frac{1}{R}[\ i\ \partial_\theta + \kappa], \tag{10}$$

with eigenvectors $e^{in\theta}$ and eigenvalues $\frac{1}{R}(n-\kappa)$. We take the radial variable to be conveniently adimensionalized throught multiplication by a fixed constant with dimensions of mass.

We will consider the action of the differential operator D on the space of functions satisfying global boundary conditions, characterized by

$$\left(\ \mathcal{P}_{\geq}\ \ \sigma(1-\mathcal{P}_{\geq})\ \sigma^*\ \right)\begin{pmatrix} \varphi(R,\theta) \\ \chi(R,\theta) \end{pmatrix} = 0, \tag{11}$$

with

$$\mathcal{P}_{\geq} = \frac{1}{2\pi} \sum_{n\geq k+1} e^{in\theta} \left\langle e^{in\theta}, \ \cdot \ \right\rangle , \tag{12}$$

where k is the integer such that $k < \kappa \leq k+1$. Notice that

$$\sigma(1-\mathcal{P}_{\geq})\sigma^* = \sigma\,\mathcal{P}_{<}\,\sigma^* = \frac{1}{2\pi} \sum_{n\leq k+1} e^{in\theta} \left\langle e^{in\theta}, \ \cdot \ \right\rangle = \mathcal{P}_{\leq} \tag{13}$$

and the operator so defined, $(D)_{\kappa}$, turns out to be self adjoint. The presence of σ as in (11) has a relevant consequence on the boundary conditions for the lower components. As we will discuss later, this also reflects on the form of the index theorem for the present situation.

Our aim is to compute the quotient of the determinants of the operators $(D)_{\kappa}$ and $(i\ \partial\!\!\!/)_{\kappa=0}$. Since the global boundary conditions in Eq. (11) depend on the flux Φ as a step function, there is no continuous family connecting both operators. So, we will proceed in two steps:

$$(D)_{\kappa} \to (i\ \partial\!\!\!/)_{\kappa} \to (i\ \partial\!\!\!/)_{\kappa=0}. \tag{14}$$

In the first step, where there is no change of boundary conditions, we can grow the gauge field by varying α from 0 to 1 in

$$D_{\alpha} = i\ \partial\!\!\!/ + \alpha\ A\!\!\!/ = e^{-\alpha\gamma_5\phi(r)}\ \ i\ \partial\!\!\!/\ e^{-\alpha\gamma_5\phi(r)},\ \text{with } 0\leq\alpha\leq 1, \tag{15}$$

thus going smoothly from the free to the full Dirac operator. The explicit knowledge of the Green's function will allow us to perform the calculation of this step, where we will use a gauge invariant point splitting regularization of the α-derivative of the determinant. The second step will be achieved by using a ζ-function regularization, after explicitly computing the spectra.

But this is not the whole story: As we are going to see, these global boundary conditions give rise to the presence of zero modes, which must be taken into account.

III. ZERO MODES

We will here show that the operator $(D_{\alpha})_{\kappa}$ has $|k+1|$ zero modes.

From (7), we get

$$D_{\alpha}\ \psi = 0 \Rightarrow \partial\!\!\!/\ e^{-\alpha\gamma_5\phi(r)}\ \psi = 0, \tag{16}$$

or, equivalently

$$\begin{pmatrix} 0 & e^{-i\theta}(\partial_r - \frac{i}{r}\partial_\theta) \\ e^{i\theta}(\partial_r + \frac{i}{r}\partial_\theta) & 0 \end{pmatrix} \begin{pmatrix} e^{-\alpha\phi(r)} & 0 \\ 0 & e^{\alpha\phi(r)} \end{pmatrix} \begin{pmatrix} \varphi(r,\theta) \\ \chi(r,\theta) \end{pmatrix} = 0. \tag{17}$$

Now, we introduce the expansions

$$\varphi(r,\theta) = \sum_{n=-\infty}^{\infty} \varphi_n(r)\ e^{in\theta},$$
$$\chi(r,\theta) = \sum_{n=-\infty}^{\infty} \chi_n(r)\ e^{in\theta}.$$

The solutions are thus given by

$$\begin{aligned} \varphi_n(r) &= \ a_n\ \ r^n\ e^{\alpha\phi(r)}, \\ \chi_n(r) &= \ b_n\ \ r^{-n}\ e^{-\alpha\phi(r)}, \end{aligned} \tag{18}$$

where the coefficients a_n and b_n are to be determined from the normalizability requirement at the origin and the boundary conditions at $r = R$, Eq. (11). Thus

$$\varphi(r,\theta) = e^{\alpha\phi(r)} \sum_{n=0}^{k} a_n \ r^n \ e^{in\theta},$$
$$\chi(r,\theta) = e^{-\alpha\phi(r)} \sum_{n=0}^{-k-2} b'_n \ r^n \ e^{-in\theta}. \tag{19}$$

Then, there are $|k+1|$ zero modes which, once normalized, are given by

$$\frac{e^{\alpha\phi(r)}}{\sqrt{2\pi \ q_n(R;\alpha)}} \begin{pmatrix} X^n \\ 0 \end{pmatrix}, \quad \text{for } 0 \leq n \leq k, \text{ if } \ k \geq 0;$$

$$\frac{e^{-\alpha\phi(r)}}{\sqrt{2\pi \ p_n(R;\alpha)}} \begin{pmatrix} 0 \\ X^{*n} \end{pmatrix}, \quad \text{for } \ 0 \leq n \leq -k-2, \text{ if } \ k < -1. \tag{20}$$

Here, the normalization factors are

$$q_n(u;\alpha) = \int_0^u e^{2\alpha\phi(r)} \ r^{2n+1} \ dr \ ,$$
$$p_n(u;\alpha) = \int_0^u e^{-2\alpha\phi(r)} \ r^{2n+1} \ dr \ . \tag{21}$$

Notice that, for $k = -1$ (in particular, when $\Phi = 0$), there is no zero mode.

So, the kernel of the orthogonal projector on $\mathrm{Ker}(D_\alpha)$ is

$$P_\alpha(z,w) = \sum_{n=0}^{k} \frac{e^{\alpha[\phi(z)+\phi(w)]}}{2\pi \ q_n(R;\alpha)} \begin{pmatrix} (ZW^*)^n & 0 \\ 0 & 0 \end{pmatrix}, \text{if } k \geq 0,$$

$$P_\alpha(z,w) = \sum_{n=0}^{-k-2} \frac{e^{-\alpha[\phi(z)+\phi(w)]}}{2\pi \ p_n(R;\alpha)} \begin{pmatrix} 0 & 0 \\ 0 & (Z^*W)^n \end{pmatrix}, \text{ if } k < -1. \tag{22}$$

Since P_α is an orthogonal projector, $P_\alpha^2 = P_\alpha$, we have

$$\partial_\alpha P_\alpha \ (1 - P_\alpha) = P_\alpha \ \partial_\alpha P_\alpha. \tag{23}$$

Being $(D_\alpha + P_\alpha)_\kappa$ invertible, we can define

$$Det^{'}(D_\alpha)_\kappa \equiv Det(D_\alpha + P_\alpha)_\kappa, \tag{24}$$

and write

$$\frac{Det^{'}(D)_\kappa}{Det(i \ \partial\!\!\!/)_{\kappa=0}} = \frac{Det(D + P_1)_\kappa}{Det(i \ \partial\!\!\!/ + P_0)_\kappa} \ \frac{Det(i \ \partial\!\!\!/ + P_0)_\kappa}{Det(i \ \partial\!\!\!/)_{\kappa=0}}. \tag{25}$$

As mentioned above, we can compute

$$\frac{\partial}{\partial\alpha} \left[\ln Det(D_\alpha + P_\alpha)_\kappa\right] = Tr \left[(A\!\!\!/ + \partial_\alpha P_\alpha) G_\alpha\right], \tag{26}$$

after determining the Green's function $G(x,y)$ satisfying

$$(D_\alpha + P_\alpha) \ G(x,y) = \delta(x,y),$$
$$\left(\mathcal{P}_{\geq} \quad \mathcal{P}_{\leq} \right) G(x,y)|_{r=R} = 0, \tag{27}$$

and applying a regularization prescription to the trace in (26). Then, by integrating in α from 0 to 1 we will get the first quotient in the r.h.s. of (25).

The computation of this Green's function will be the subject of the next section.

IV. CALCULATION OF THE GREEN'S FUNCTION

Notice that, since D_α is self adjoint, we can write

$$G(x,y) = (1-P_\alpha)\ \mathcal{G}(x,y)\ (1-P_\alpha) + \ P_\alpha, \tag{28}$$

where $\mathcal{G}(x,y)$ is the kernel of the right-inverse of D_α on the orthogonal complement of $Ker(D_\alpha)$. From the Eq. (7), it is easy to see that

$$\mathcal{G}(x,y) = e^{\alpha\gamma_5\phi(r)}\ \mathcal{G}_0(x,y)\ e^{\alpha\gamma_5\phi(r')}, \tag{29}$$

where $\mathcal{G}_0(x,y)$ can be obtained by solving the differential equation

$$i\ \partial\!\!\!/ \begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix} f\\ g\end{pmatrix}, \tag{30}$$

for $\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$ satisfying

$$P_\alpha\ \ e^{\alpha\gamma_5\phi(r)}\ \begin{pmatrix}\varphi\\ \chi\end{pmatrix} = 0,$$

$$\left(\ \mathcal{P}_{\geq}\quad \mathcal{P}_{\leq}\ \right) e^{\alpha\gamma_5\phi(R)} \begin{pmatrix}\varphi\\ \chi\end{pmatrix} = 0. \tag{31}$$

Now, by expanding

$$\varphi(r,\theta) = \sum \varphi_n(r)\ e^{in\theta},\ \ \chi(r,\theta) = \sum \chi_n(r)\ e^{in\theta}$$

$$f(r,\theta) = \sum f_n(r)\ e^{in\theta},\ \ g(r,\theta) = \sum g_n(r)\ e^{in\theta}, \tag{32}$$

we get

$$\varphi(r,\theta) = i \sum_{n\geq k+1} e^{in\theta} r^n \int_r^R dr'\ (r')^{-n}\ g_{n+1}(r') -\ i\sum_{n\leq k} e^{in\theta} r^n \int_0^r dr'\ (r')^{-n}\ g_{n+1}(r')$$

$$= \frac{i}{2\pi}\int_r^R dr' \int_0^{2\pi} d\theta' \sum_{n\geq k+1} e^{in(\theta-\theta')} \left(\frac{r}{r'}\right)^n e^{-i\theta'} g(r',\theta') - \frac{i}{2\pi}\int_0^r dr' \int_0^{2\pi} d\theta' \sum_{n\leq k} e^{in(\theta-\theta')} \left(\frac{r}{r'}\right)^n e^{-i\theta'} g(r',\theta'). \tag{33}$$

Analogously

$$\chi(r,\theta) = \frac{i}{2\pi}\int_r^R dr' \int_0^{2\pi} d\theta' \sum_{n\leq k+1} e^{in(\theta-\theta')} \left(\frac{r'}{r}\right)^n e^{i\theta'} f(r',\theta') - \frac{i}{2\pi}\int_0^r dr' \int_0^{2\pi} d\theta' \sum_{n\geq k+2} e^{in(\theta-\theta')} \left(\frac{r'}{r}\right)^n e^{i\theta'} f(r',\theta'). \tag{34}$$

Therefore, we have

$$\mathcal{G}_0(x,y) = \frac{1}{2\pi i}\begin{pmatrix} 0 & \frac{1}{X-Y}\left(\frac{X}{Y}\right)^{k+1} \\ \frac{1}{X^*-Y^*}\left(\frac{Y^*}{X^*}\right)^{k+1} & 0\end{pmatrix}, \tag{35}$$

and

$$\mathcal{G}(x,y) = \frac{1}{2\pi i}\begin{pmatrix} 0 & \frac{e^{\alpha[\phi(x)-\phi(y)]}}{X-Y}\left(\frac{X}{Y}\right)^{k+1} \\ \frac{e^{-\alpha[\phi(x)-\phi(y)]}}{X^*-Y^*}\left(\frac{Y^*}{X^*}\right)^{k+1} & 0\end{pmatrix}. \tag{36}$$

Finally, from Eq. (28), after a straightforward but tedious computation, we obtain, for $k\geq 0$,

$$G(x,y)=\frac{1}{2\pi i}\begin{pmatrix} ie^{\alpha[\phi(x)+\phi(y)]}\sum\limits_{n=0}^{k}\frac{(XY^*)^n}{q_{n(R;\alpha)}} & e^{\alpha[\phi(x)-\phi(y)]}\left[\frac{1}{X-Y}+\frac{1}{Y}\sum\limits_{n=0}^{k}\left(\frac{X}{Y}\right)^n\frac{q_{n(y;\alpha)}}{q_{n(R;\alpha)}}\right] \\ e^{-\alpha[\phi(x)-\phi(y)]}\left[\frac{1}{X^*-Y^*}-\frac{1}{X^*}\sum\limits_{n=0}^{k}\left(\frac{Y^*}{X^*}\right)^n\frac{q_{n(x;\alpha)}}{q_{n(R;\alpha)}}\right] & 0\end{pmatrix}, \tag{37}$$

and, for $k<-1$,

$$G(x,y)=\frac{1}{2\pi i}\begin{pmatrix} 0 & e^{\alpha[\phi(x)-\phi(y)]}\left[\frac{1}{X-Y}-\frac{1}{X}\sum\limits_{n=0}^{-k-2}\left(\frac{Y}{X}\right)^n\frac{p_{n(x;\alpha)}}{p_{n(R;\alpha)}}\right] \\ e^{-\alpha[\phi(x)-\phi(y)]}\left[\frac{1}{X^*-Y^*}+\frac{1}{Y^*}\sum\limits_{n=0}^{-k-2}\left(\frac{X^*}{Y^*}\right)^n\frac{p_{n(y;\alpha)}}{p_{n(R;\alpha)}}\right] & ie^{-\alpha[\phi(x)+\phi(y)]}\sum\limits_{n=0}^{-k-2}\frac{(X^*Y)^n}{p_{n(R;\alpha)}}\end{pmatrix}. \tag{38}$$

V. EVALUATION OF THE DETERMINANT

Now we have all the necessary elements to compute the first quotient in the r.h.s. of Eq. (25). In fact, from Eqs. (28) and (23)

$$Tr\left[(\partial_\alpha P_\alpha)\,G_\alpha\right]=0. \tag{39}$$

So, Eq. (26) becomes

$$\frac{\partial}{\partial\alpha}\left[\ln Det(D_\alpha+P_\alpha)_\kappa\right]=Tr\left[\not{A}G_\alpha\right]. \tag{40}$$

As usual, the kernel of the operator inside the trace is singular at the diagonal, so we must introduce a regularization. We will employ a point-splitting one where, following Schwinger [17], we will introduce a phase factor in order to preserve gauge invariance. We thus get, for $k\geq 0$, from (37)

$$\begin{aligned} Tr\left[\not{A}G_\alpha\right] &= \int_{r<R} d^2x\; tr\left[\not{A}(x)G_\alpha(x,x+\epsilon)e^{i\alpha\epsilon\cdot A(x)}\right]_{\epsilon\to 0} \\ &= -\frac{1}{2\pi}\int_{r<R} d^2x\,\phi'(r)\left\{e^{-i\theta}e^{\alpha\phi'(r)[\epsilon_r-i\epsilon_\theta]}\left[\frac{e^{i\theta}}{\epsilon_r-i\epsilon_\theta}+\frac{1}{X^*}\sum_{n=0}^{k}\frac{q_{n(x;\alpha)}}{q_{n(R;\alpha)}}+O(\epsilon)\right]\right. \\ &\qquad \left.+\,e^{i\theta}e^{-\alpha\phi'(r)[\epsilon_r+i\epsilon_\theta]}\left[-\frac{e^{-i\theta}}{\epsilon_r+i\epsilon_\theta}+\frac{1}{X}\sum_{n=0}^{k}\frac{q_{n(x;\alpha)}}{q_{n(R;\alpha)}}+O(\epsilon)\right]\right\}_{\epsilon\to 0} \\ &= -\left[\frac{2i\epsilon_\theta}{\epsilon_r^2+\epsilon_\theta^2}\right]_{\epsilon\to 0}\int_0^R dr\,r\phi'(r)-\frac{\alpha}{\pi}\int_{r<R}d^2x\;{\phi'}^2-2\;(k+1)\;\phi(R)\;+\sum_{n=0}^{k}\;\frac{\partial}{\partial\alpha}\ln\left[q_n(R;\alpha)\right]. \end{aligned} \tag{41}$$

Finally, performing a "symmetric" $\epsilon\to 0$ limit, which drops out the first term, and integrating in α from 0 to 1, we get

$$\ln\left[\frac{Det(D+P_1)_\kappa}{Det(i\not{\partial}+P_0)_\kappa}\right]=-\frac{1}{2\pi}\int_{r<R}d^2x\;{\phi'}^2-2\;(k+1)\;\phi(R)\;+\sum_{n=0}^{k}\;\ln\left[2(n+1)\frac{q_n(R;1)}{R^{2(n+1)}}\right]. \tag{42}$$

When there are no zero modes ($k+1=0$) only the first term in the r.h.s. survives. For $k<-1$, from (38), one gets a similar expression where the sum runs from 0 to $-k-2=|k+1|-1$, and the norms, defined in Eq. (21), q_n, are replaced by p_n. This result can also be obtained by performing the change $\phi\to-\phi$ and $k+1\to-(k+1)$. Moreover, as expected from (7), this expression is invariant under the shift $\phi(r)\to\phi(r)+\epsilon$, for any constant ϵ.

In the following, we will obtain the second quotient of determinants in Eq. (25) by computing explicitly the spectra of the free Dirac operators and using a ζ-function regularization. The equation to be solved is

$$i\ \partial\!\!\!/\psi = \lambda\psi, \tag{43}$$

or, explicitly

$$i \begin{pmatrix} 0 & e^{-i\theta}(\partial_r - \frac{i}{r}\partial_\theta) \\ e^{i\theta}(\partial_r + \frac{i}{r}\partial_\theta) & 0 \end{pmatrix} \begin{pmatrix} \varphi(r,\theta) \\ \chi(r,\theta) \end{pmatrix} = \lambda \begin{pmatrix} \varphi(r,\theta) \\ \chi(r,\theta) \end{pmatrix}. \tag{44}$$

Now, by expanding

$$\begin{aligned} \varphi(r,\theta) &= \sum_{n=-\infty}^{\infty} \varphi_n(r)\ e^{in\theta}, \\ \chi(r,\theta) &= \sum_{n=-\infty}^{\infty} \chi_n(r)\ e^{in\theta}, \end{aligned} \tag{45}$$

we get

$$\begin{aligned} i\ \varphi_n'(z) - i\ \frac{n}{z}\varphi_n(z) &= \frac{\lambda}{|\lambda|}\ \chi_{n+1}(z) \\ i\ \chi_{n+1}'(z) + i\ \frac{(n+1)}{z}\ \chi_{n+1}(z) &= \frac{\lambda}{|\lambda|}\ \varphi_n(z) \end{aligned} \tag{46}$$

where $z = |\lambda| r$. So, $\varphi_n(z)$ satisfies

$$\varphi_n''(z) + \frac{1}{z}\varphi_n'(z) + (1 - \frac{n^2}{z^2})\varphi_n(z) = 0. \tag{47}$$

Therefore, the general solution of Eq. (44) is

$$\psi(r,H) = \sum_{n=0}^{\infty} \psi_n \begin{pmatrix} J_n(|\lambda| r)\ e^{in\theta} \\ -i\frac{|\lambda|}{\lambda} J_{n+1}(|\lambda| r)\ e^{i(n+1)\theta} \end{pmatrix}. \tag{48}$$

Now, by imposing the global boundary conditions given in Eq. (11), it follows that: If $n \geq k+1$, the upper component must vanish at the boundary, which implies $\lambda = \pm j_{n,l}/R$ ($j_{n,l}$ is the l-th zero of $J_n(z)$). Analogously, if $n \leq k$, it is the lower component which must vanish, and so $\lambda = \pm j_{n+1,l}/R$. Therefore, the eigenvalues are

$$\lambda_{n,l} = \pm j_{n,l}/R,\ \text{for } n = 0, \pm 1, \pm 2, ... \text{ and } \quad l = 1, 2, ... \ . \tag{49}$$

Notice that $j_{-n,l} = j_{n,l}$, and that, for $n = k+1$ the eigenvalues appear twice, once for an eigenfunction with vanishing upper component at the boundary, and once for another one with vanishing lower component.

From these eigenvalues we can construct the ζ-function

$$\zeta_{(i\partial\!\!\!/+P_0)_\mathcal{K}}(s) = |k+1| + (1+e^{-i\pi s}) \left\{ \sum_{n=-\infty}^{n=\infty} \sum_{l=1}^{\infty} \left(\frac{j_{n,l}}{R}\right)^{-s} + \sum_{l=1}^{\infty} \left(\frac{j_{|k+1|,l}}{R}\right)^{-s} \right\}. \tag{50}$$

The first term, $|k+1|$, is the multiplicity of the 0-eigenvalue of $(i\ \partial\!\!\!/)_\mathcal{K}$. It is also interesting to note that the double sum in the r.h.s. corresponds to the ζ-function of the Laplacian on a disk with Dirichlet boundary conditions, thus being analytic at $s=0$ [12] . The second sum, to be explicitly computed below, is regular at $s=0$. Then $\zeta_{(i\partial\!\!\!/+P_0)_\mathcal{K}}(s)$ is regular at the origin. As far us we know, the ζ-regularity at the origin for non local boundary conditions has not been established in general [18].

From (50), we can write

$$\begin{aligned} \ln\left[\frac{Det(i\ \partial\!\!\!/ + P_0)_\mathcal{K}}{Det(i\ \partial\!\!\!/)_{\mathcal{K}=0}}\right] &= -\frac{d}{ds}\left[\zeta_{(i\partial\!\!\!/+P_0)_\mathcal{K}}(s) - \zeta_{(i\partial\!\!\!/)_{\mathcal{K}=0}}(s)\right]_{s=0} \\ = -\frac{d}{ds}&\left[(1+e^{-i\pi s})\ R^s \left\{\sum_{l=1}^{\infty}(\ j_{|k+1|,l})^{-s} - \sum_{l=1}^{\infty}\ (j_{0,l})^{-s}\right\}\right]_{s=0} \\ &= -2\left[f_{|k+1|}'(0) - f_0'(0) + (\ln R - \frac{i\pi}{2})[f_{|k+1|}(0) - f_0(0)]\right], \end{aligned} \tag{51}$$

where the function f is defined as

$$f_\nu(s) \equiv \sum_{l=1}^{\infty} (\ j_{\nu,l})^{-s}. \tag{52}$$

Taking into account the asymptotic expansion for the zeros of Bessel functions [19]

$$(\ j_{\nu,l})^{-s} - (l\pi)^{-s} + s\left(\frac{2\nu-1}{4}\right)\pi(l\pi)^{-s-1} \sim O(l^{-s-2}), \tag{53}$$

we can write

$$f_\nu(s) = \sum_{l=1}^{\infty}\left[(\ j_{\nu,l})^{-s} - (l\pi)^{-s} + s\left(\frac{2\nu-1}{4}\right)\pi(l\pi)^{-s-1}\right] + \ \pi^{-s}\left[\zeta(s) - s\left(\frac{2\nu-1}{4}\right)\zeta(s+1)\right], \tag{54}$$

where $\zeta(s)$ is Riemann's ζ-function. Notice that the first series converges for $Re\ s > -1$; thus, it can be evaluated at $s = 0$, to obtain

$$f_\nu(0) = -\frac{\nu}{2} - \frac{1}{4}, \tag{55}$$

and

$$f'_\nu(0) = -\frac{1}{2}\ln 2 + \left(\frac{2\nu-1}{4}\right)(\ln\pi - \gamma) - \sum_{l=1}^{\infty}\ln\left[\frac{j_{\nu,l}}{l\pi}\ e^{-\left(\frac{2\nu-1}{4\ l}\right)}\right], \tag{56}$$

where γ is Euler's constant. By replacing Eqs. (55) and (56) into Eq. (51), we get

$$\ln\left[\frac{Det(i\ \not\partial + P_0)_\mathcal{K}}{Det(i\ \not\partial)_{\mathcal{K}=0}}\right] = -|k+1|[\frac{i\pi}{2} - \gamma - \ln(\frac{R}{\pi})] + 2\sum_{l=1}^{\infty}\ln\left[\frac{j_{|k+1|,l}}{j_{0,l}}\ e^{-\left(\frac{|k+1|}{2\ l}\right)}\right]. \tag{57}$$

Finally, adding (42) and (57), and taking into account that we have used a gauge invariant procedure, we obtain for the quotient in the l.h.s. of (25)

$$\ln\left[\frac{Det(D + P_1)_\mathcal{K}}{Det(i\ \not\partial)_{\mathcal{K}=0}}\right] = -\frac{1}{2\pi}\int_{r<R} d^2x\ A_\mu(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2})\ A_\nu - 2\ (k+1)\ \phi(R)$$

$$+\sum_{n=0}^{k}\ \ln\left[2(n+1)\frac{q_n(R;1)}{R^{2(n+1)}}\right] - |k+1|[\frac{i\pi}{2} - \gamma - \ln(\frac{R}{\pi})] + 2\sum_{l=1}^{\infty}\ln\left[\frac{j_{|k+1|,l}}{j_{0,l}}\ e^{-\left(\frac{|k+1|}{2\ l}\right)}\right]. \tag{58}$$

The first term is the integral on the disk of the same expression appearing in the well-known result for the boundaryless case [20]. Note that this final result is also invariant under the transformation: $\phi \rightarrow \phi + \epsilon$ for any constant ϵ.

Now, a comment is in order concerning global axial transformations and their relationship to zero modes. Under such a transformation, with constant ϵ,

$$e^{-\gamma_5\epsilon}(i\ \not\partial)_{\mathcal{K}=0}e^{-\gamma_5\epsilon} = (i\ \not\partial)_{\mathcal{K}=0} \tag{59}$$

is invariant. Moreover

$$e^{-\gamma_5\epsilon}(D + P_1)_\mathcal{K}e^{-\gamma_5\epsilon} = (D + e^{-\gamma_5\epsilon}P_1e^{-\gamma_5\epsilon})_\mathcal{K}, \tag{60}$$

while its inverse has the structure

$$G^{(\epsilon)}(x,y) = (1-P_1)\ \mathcal{G}(x,y)\ (1-P_1) + e^{\gamma_5\epsilon}\ P_1\, e^{\gamma_5\epsilon}. \tag{61}$$

Therefore, since γ_5 leaves Ker D invariant,

$$\frac{\partial}{\partial\epsilon}\ln\left[\frac{Det\,(e^{-\gamma_5\epsilon}(D+P_1)_\kappa e^{-\gamma_5\epsilon})}{Det\,(e^{-\gamma_5\epsilon}(i\not\partial)_{\kappa=0}e^{-\gamma_5\epsilon})}\right]=-Tr\left[e^{-\gamma_5\epsilon}\{\gamma_5,P_1\}\,e^{-\gamma_5\epsilon}G^{(\epsilon)}\right]=-2Tr\left[\gamma_5 P_1\right]=-2(N_+-N_-),\tag{62}$$

where $N_{+(-)}$ is the number of positive(negative) chirality zero modes.

It can be verified that our strategy leads to the correct result for the index of D. By following the same procedure that lead to Eq. (58), we can compute the quotient of determinants in the l.h.s of (62). In fact, using Eq. (61) instead of Eq. (28), the only difference appears in the first term of the r.h.s. of (50), where a factor $e^{\pm 2\epsilon s}$ arises. Thus, after performing the ϵ-derivative

$$N_+-N_-=k+1,\tag{63}$$

which agrees with our previous result for the number of zero modes.

For the sake of completeness, in the next section we will show how the presence of the σ's in (7) and (11) affects the index theorem.

VI. RELATION TO THE INDEX THEOREM

Here we will reobtain the index by following the Atiyah-Patodi-Singer approach [13].

The Dirac operator in polar coordinates can be written as [15,16]

$$D=\frac{1}{\sqrt{r}}\begin{pmatrix}0 & e^{-i\theta/2}\,(\partial_r+B)e^{-i\theta/2}\\ e^{i\theta/2}\,(-\partial_r+B)e^{i\theta/2} & 0\end{pmatrix}\sqrt{r},\tag{64}$$

subject to the boundary conditions in Eq. (11). It is to be stressed that the presence of the factors $e^{\pm i\theta/2}$ coming from the σ's will produce a departure from the index formula in [13].

To be be consistent with (11) we call $\mathcal{D}$ the differential operator $(-\partial_r+B)$ acting on functions

$$f(r,\theta)=\sum_i^n f_n(r)e^{i(n+\frac{1}{2})\theta}\tag{65}$$

satisfying

$$\mathcal{P}_{\geq}\left[e^{-i\theta/2}f(R,\theta)\right]=0.\tag{66}$$

Its adjoint, $\mathcal{D}^+$, is the differential operator (∂_r+B) defined on the space of functions

$$g(r,\theta)=\sum_i^n g_n(r)e^{i(n-\frac{1}{2})\theta}\tag{67}$$

satisfying

$$\mathcal{P}_{\leq}\left[e^{i\theta/2}g(R,\theta)\right]=0.\tag{68}$$

Then $D^+D=\frac{1}{\sqrt{r}}e^{-i\theta/2}\,\mathcal{D}^+\mathcal{D}\,e^{i\theta/2}\sqrt{r}$, where $\mathcal{D}^+\mathcal{D}=-\partial_r^2+B^2$ acting on functions satisfying

$$\begin{cases}\mathcal{P}_{\geq}\left[e^{-i\theta/2}f(R,\theta)\right]=0\\ \mathcal{P}_{\leq}\left[e^{i\theta/2}\mathcal{D}f(R,\theta)\right]=0.\end{cases}\tag{69}$$

or, equivalently,

$$\begin{cases}f_n(R)=0 & \text{for } n\geq k+1\\ -f_n'(R)+\dfrac{n+1/2-\kappa}{R}f_n(R)=0 & \text{for } n\leq k.\end{cases}\tag{70}$$

Similarly, $\mathcal{D}\mathcal{D}^+ = -\partial_r^2 + B^2$ acting on functions $g(r,\theta)$ satisfying

$$\begin{cases} g_n(R) = 0 & \text{for } n \leq k+1 \\ g_n'(R) + \dfrac{n+1/2-\kappa}{R} g_n(R) = 0 & \text{for } n \geq k+2. \end{cases} \tag{71}$$

Now, we have for the heat kernels

$$Tr\left\{e^{-tD^+D}\right\} - Tr\left\{e^{-tDD^+}\right\} = Tr\left\{e^{-t\mathcal{D}^+\mathcal{D}}\right\} - Tr\left\{e^{-t\mathcal{D}\mathcal{D}^+}\right\}. \tag{72}$$

When written in this fashion, the boundary contribution to the r.h.s. in (72), $K(t)$, can be easily constructed from Eqs. (2.16) and (2.17) in [13], thus getting

$$K(t) = \frac{1}{2}\,\mathrm{erfc}\left(-(k+\frac{1}{2}-\kappa)\sqrt{t}\right) - \sum_{n\neq k} \frac{sig(n+\frac{1}{2}-\kappa)}{2}\,\mathrm{erfc}\left(|\,n+\frac{1}{2}-\kappa|\,\sqrt{t}\right). \tag{73}$$

Taking into account that

$$\lim_{t\to\infty} K(t) = m(\kappa) = \begin{cases} 0 & \text{for} \quad k+\frac{1}{2} < \kappa \leq k+1 \\ \frac{1}{2} & \text{for} \quad \kappa = k+\frac{1}{2} \\ 1 & \text{for} \quad k < \kappa < k+\frac{1}{2} \end{cases}, \tag{74}$$

we obtain

$$\int_0^\infty dt\; t^{s-1}\left\{K(t) - m(\kappa)\right\} = \frac{-\Gamma(s+1/2)}{2s\sqrt{\pi}}\;\eta_{(B+\frac{1}{2R})}(2s), \tag{75}$$

where

$$\eta_{(B+\frac{1}{2R})}(s) = R^s \sum_{n\neq\kappa-\frac{1}{2}} sig(n+\frac{1}{2}-\kappa)\;|n+\frac{1}{2}-\kappa|^{-s}. \tag{76}$$

In particular

$$\eta_{(B+\frac{1}{2R})}(0) = \begin{cases} 2(\kappa-k-1) & \text{for} \quad k+\frac{1}{2} < \kappa \leq k+1 \\ 0 & \text{for} \quad \kappa = k+\frac{1}{2} \\ 2(\kappa-k) & \text{for} \quad k < \kappa < k+\frac{1}{2} \end{cases}. \tag{77}$$

Finally, taking into account that, for $k < \kappa \leq k+1$,

$$m(\kappa) - \frac{1}{2}\,\eta_{(B+\frac{1}{2R})}(0) = k+1-\kappa = \frac{1}{2}\left[1 - h(B) - \eta_{(B)}(0)\right], \tag{78}$$

where $h(B) = \dim\ker(B)$, and from the asymptotic expansion [13] of the heat kernels in (72), we have

$$\mathrm{index}\,D = \kappa + \frac{\left[1-h(B)-\eta_{(B)}(0)\right]}{2} = k+1, \tag{79}$$

in agreement with (63). The first term in the intermediate expression is the well known contribution from the bulk [20]. The second one is the boundary contribution of APS, shifted by 1/2. This correction, due to the presence of the factor σ in (7), has already been obtained in [15] with slightly different spectral boundary conditions.

VII. CONCLUSIONS

In this paper we have achieved the complete evaluation of the determinant of the Dirac operator on a disk, in the presence of an axially symmetric flux, under global boundary conditions of the type intoduced by Atiyah, Patodi and Singer [13]. To this end, we have proceeded in two steps: In the first place, we have grown the gauge field while keeping the boundary condition fixed. This calculation was possible thanks to the exact knowledge of the zero modes

and the Green's function (in the complement of the null space.) Here, a gauge invariant point splitting regularization was employed. In references [10,11] we developed, for the case of local boundary conditions, a regularization scheme based on Seeley's complex powers [12]. Its application to the present problem leads to the same result as the point splitting we used here, even though its relation to the ζ-function is not guaranteed. In fact, as far as we know, the construction of complex powers for elliptic boundary problems with global boundary conditions is still under study [18].

In the second step, we have explicitly obtained the eigenvalues of $(i\,\partial\!\!\!/+P_0)_\kappa$. We have shown that the corresponding ζ-function is regular at the origin and we have evaluated the quotient of the free Dirac operators for two different global boundary conditions.

We have verified that our complete result is in agreement with the APS index theorem, adapted to our example.

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